MOTION IN TWO AND THREE DIMENSIONS

Position vector : A straight line directed from the origin O to any point A at time t is known as the position vector of the moving particle at the time instant t.

Displacement vector : The displacement vector of a moving particle in a given time interval is a segment (→) obtained by taking the initial position of the moving particle as the tail and its final position as the tip of the '→' (arrow). Corresponding, displacement represents the change in position of the moving particle in the given time interval.

Vector and scalar quantities : Physical quantities which require direction as well as magnitude for their complete specification, are known as vectors or vector quantities, e.g., velocity, force, momentum, displacement, acceleration, etc.

Physical quantities which are completely specified by their magnitude alone are known as scalars or scalar quantities. e.g., distance, mass, speed, temperature, volume, etc.

Difference between Scalars and Vectors:

Scalars	Vectors
• Scalars are specified by magnitude only.	• Vectors require magnitude as well as direction to specify them.
• These can be added or subtracted according to ordinary rule of algebra.	• These can not be added according to ordinary rule of algebra.
• Scalars change by change in magnitude only.	• Vectors change with change in either magnitude or direction or both.
• These are represented by ordinary letters, e.g., AB, BC, CD.	• These are represented by bold font letters or letters having arrow heads over e.g., **AB**, **BC**, **CD** or $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$

Kinds of Vector

(i) **Equal vectors :** Two vectors are said to be equal if and only if they have same magnitude and direction.

(ii) **Unequal vectors :** Two vectors are said to be unequal when either magnitudes of the two vectors are not the same, or their directions are not parallel.

(iii) **Unit vector :** A vector divided by its magnitude is called a unit vector, *i.e.*,

$$\vec{A} = \frac{\vec{A}}{|\vec{A}|}$$

The three rectangular unit vectors along three coordinate axes x, y and z are represented by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively.

(iv) **Zero vector or Null vector :** It is a vector whose magnitude is zero as its length is zero. There is no need to specify the direction of this vector. Hence $|\vec{0}| = 0$.

(v) **Collinear vectors :** Vectors having the same or parallel directions are said as collinear vectors.

(vi) **Coplanar vectors :** A system of vectors are said to be coplanar if their supports are parallel to the same plane.

(vii) **Negative vector :** A vector is said to be negative of a given vector if its magnitude is the same as that of the given vector but direction is reverse.

(viii) **Fixed vector :** Fixed vector is that whose initial point or tail is fixed.

(ix) **Free vector :** Free vector is that whose initial point or tail is not fixed.

(x) **Co-initial vectors :** These are those which have the same initial point.

(xi) **Co-terminus vectors :** These are those which have same terminal point.

Laws of Vector Addition

(i) **Parallelogram law of vectors :** It states that when two vectors are represented in magnitude and direction by two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal passing through the point of intersection of these two sides.

(ii) **Triangle law of vectors :** It states that if two given vectors are represented in

magnitude and direction by the two sides of a triangle taken in order, then the third side taken in opposite order gives the resultant vector in magnitude and direction.

(iii) **Polygon law of vectors :** It states that when a number of vectors are represented in magnitude and direction by the sides of a polygon taken in order, then the resultant vector is given in magnitude and direction by the closing side taken in opposite order.

Commutative law of vector addition :

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

Associative law of vector addition :

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

Distributive law of vector addition :

$$\lambda(\vec{A} + \vec{B}) = \lambda\vec{A} + \lambda\vec{B}$$

Resolution of a vector : The mechanism of splitting a vector into two or more vectors along given directions which if applied together, will produce exactly the same effect as the given single vector does, is called as resolution of a given vector and the parts so obtained are called as the *components of the vector*.

Rectangular components of a vector : If the components of a given vector are perpendicular to each other, then they are called as rectangular components.

If $\vec{A}_x$ and $\vec{A}_y$ are two rectangular components of a vector $\vec{A}$, then

$$\vec{A} = \vec{A}_x + \vec{A}_y = x\hat{i} + y\hat{j}$$

$$\text{and } A = \sqrt{A_x^2 + A_y^2} = \sqrt{x^2 + y^2}$$

$$\text{and } \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) = \tan^{-1}\left(\frac{y}{x}\right)$$

In three dimensions :

$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{and } A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{x^2 + y^2 + z^2}$$

where,

$$A = |\vec{A}|$$

$$A_x = |\vec{A}_x| = x = A\cos\theta$$

$$A_y = |\vec{A}_y| = y = A\sin\theta$$

$$A_z = |\vec{A}_z| = z$$

Direction cosine of a vector

The direction cosines l, m and n of a vector are the cosines of the angles α, β and γ which a given vector makes with x-axis, y-axis and z-axis respectively, i.e.,

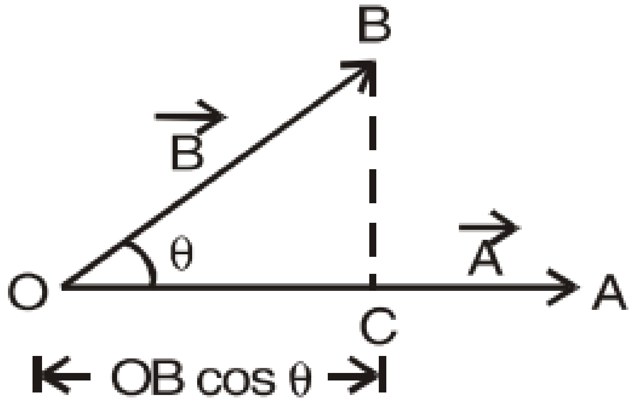
$$l = \cos\alpha = \frac{A_x}{A}$$

$$m = \cos\beta = \frac{A_y}{A}$$

$$n = \cos\gamma = \frac{A_z}{A}$$

and $l^2 + m^2 + n^2 = \dfrac{A_x^2 + A_y^2 + A_z^2}{A^2} = \dfrac{A^2}{A^2} = 1$

Scalar product or dot product : Scalar product of the two vectors is the product of the size of one vector and the component of the second in the direction of the first.



Hence $\vec{A}.\vec{B} = |\vec{A}||\vec{B}| \cos\theta$

Hence, scalar product is also explained as the product of the magnitudes of the two vectors and cosine of the angle between their directions.

Properties :

(i) Dot product of two vectors is commutative, i.e.,

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

(ii) Dot product is distributive, i.e.,

$$\vec{A}.(\vec{B} + \vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$$

(iii) Dot product of two mutually perpendiculars vectors is zero. For unit vectors,

$$\hat{i}.\hat{j} = 0, \quad \hat{j}.\hat{k} = 0, \quad \hat{k}.\hat{i} = 0$$

(iv) Dot product of two equal vectors is equal to the square of the magnitude of either of the two vectors. For unit vectors,

$$\hat{i}.\hat{i} = 1, \quad \hat{j}.\hat{j} = 1, \quad \hat{k}.\hat{k} = 1$$

(v) Dot product of two collinear vectors is equal to the positive or the negative of the product of their magnitudes.

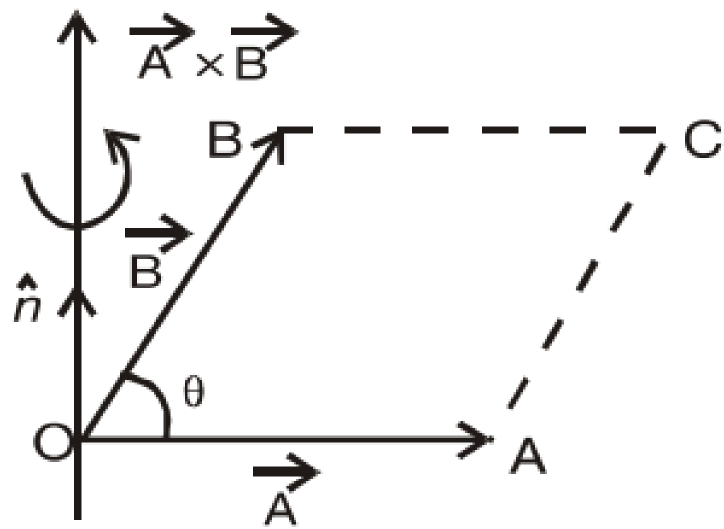
Applications :

(i) Work done (W) by a force is defined as the dot product of force $(\vec{F})$ and the displacement produced $(\vec{S})$. *i.e.,* $W = \vec{F}.\vec{S}$.

(*ii*) Power (P) is defined as the dot product of force ($\vec{F}$) and velocity ($\vec{V}$). *i.e.,* $P = \vec{F}.\vec{V}$.

(*iii*) Magnetic Flux (ϕ) is defined as the dot product of magnetic induction ($\vec{B}$) and area vector ($\vec{A}$). *i.e.,* $\phi = \vec{B}.\vec{A}$.

Vector product or Cross product : Vector product of the two vectors is defined as a vector having a magnitude equal to the product of the magnitudes of the vectors and the sine of the angle between their directions, and a direction perpendicular to the plane containing both the vectors. Hence



$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta.\hat{n}$$

Also, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

Properties :

(*i*) Cross product of two vectors is not commutative, i.e.,

$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

For unit vectors,

$$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i}$$

$$\hat{j} \times \hat{k} = -\hat{k} \times \hat{j}$$

$$\hat{k} \times \hat{i} = -\hat{i} \times \hat{k}$$

(ii) Cross product is distributive, i.e.,

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(iii) Cross product of two mutually perpendicular vectors is equal to the product of the magnitudes of the vectors in the direction perpendicular to the plane of the two vectors.

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$$

(iv) If two vectors represent the adjacent sides of a parallelogram, then magnitude of cross product of these vectors represents the area of the parallelogram.

(v) If two vectors represent two sides of a triangle, then half the magnitude of cross product of these vectors represents the area of the triangle.

Applications :

(i) Instantaneous velocity $(\vec{v})$ of a particle is equal to the cross product of its angular velocity $(\vec{\omega})$ and position vector $(\vec{r})$, i.e.,

$$\vec{v} = \vec{\omega} \times \vec{r}$$

(ii) Tangential acceleration ($\vec{a}$) of a particle is equal to the cross product of its angular acceleration ($\vec{\alpha}$) and position vector ($\vec{r}$). i.e.,

$$\vec{a} = \vec{\alpha} \times \vec{r}$$

(iii) Angular momentum ($\vec{L}$) of a particle is equal to the cross product of its position vector ($\vec{r}$) and linear momentum ($\vec{p}$). i.e.,

$$\vec{L} = \vec{r} \times \vec{p}$$

(iv) Torque ($\vec{\tau}$) of a force acting on a particle is equal to the cross product of its position vector ($\vec{r}$) and force applied ($\vec{F}$). i.e.,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Uniform velocity : If a particle covers equal displacements in equal intervals of time, it is said to be moving with uniform velocity. Hence,

$$\text{Velocity} = \frac{\text{displacement}}{\text{time interval}}$$

$$\text{or} \qquad \vec{v} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}$$

Velocity, in terms of rectangular components, is given by,

$$\vec{v} = v_x\hat{i} + v_y\hat{j} \qquad \text{(in two dimensions)}$$

and $$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \quad \text{(in three dimensions)}$$

Uniform acceleration : If the rate of change of velocity of a particle with time is constant, it is said to be moving with a uniform acceleration. Hence,

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{time interval}}$$

$$\text{or } \vec{a} = \frac{\vec{v}(t_2) - \vec{v}(t_1)}{t_2 - t_1}$$

Acceleration, in terms of rectangular components, is given by,

$$\vec{a} = a_x\hat{i} + a_y\hat{j} \qquad \text{(in two dimensions)}$$

and $$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \quad \text{(in three dimensions)}$$

Uniform circular motion : When an object moves along a circular path with uniform speed, its motion is called uniform circular motion.

(i) **Angular velocity :** It is the rate at which angle swept by the radius at the centre changes with time.

If is also defined as the rate of change of angular displacement. It is represented by ω. Hence,

$$\text{Angular velocity} = \frac{\text{angular displacement}}{\text{time elapses}}$$

$$\text{or } \omega = \frac{\theta}{t}$$

Instantaneous angular velocity is given by,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

And its unit is rad s^{-1}.

(ii) **Relation between linear velocity and Angular velocity :** Linear velocity of a particle is the product of its angular velocity and the radius of the circle in which it is rotating its direction being tangential to the circle at the point. i.e., $v = r\omega$.

(iii) **Angular acceleration :** Rate of change of angular velocity is called as angular acceleration. It is represented by α.

$$\text{Hence } \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

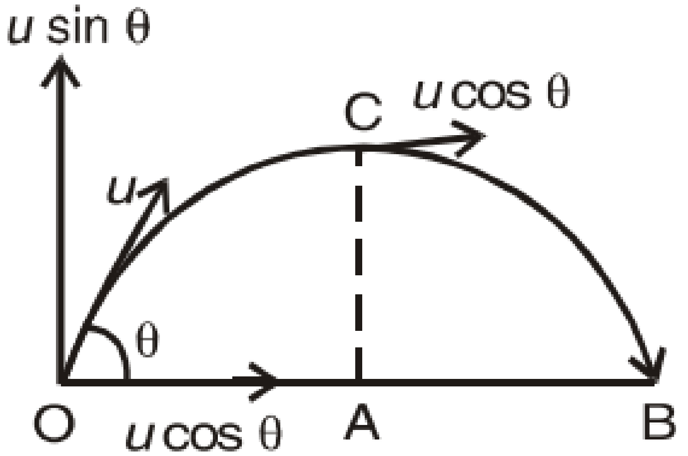
Its unit is rad s^{-2}.

Projectile : A body thrown at a certain angle with the horizontal is called a projectile.

Projectile motion : When a particle is so projected, that it makes certain angle with the horizontal, then the motion of the particle is said to be projectile motion.

Important :

- Path of the projectile is a parabola.



- The initial velocity u of the projectile can be resolved into two components :
  (a) $u \cos \theta \rightarrow$ horizontal direction
  (b) $u \sin \theta \rightarrow$ vertical direction.
- Since, there is no acceleration in the horizontal direction, the horizontal component of velocity remains the same throughout the motion.
- The particle returns to ground at the same angle and with the same velocity with which it was projected.
- The vertical component of velocity at the highest point of the path is zero. Only the

horizontal component $u \cos \theta$ remains at the highest point of path.

Time of flight : It is the total time taken by the projectile from the point of projection till it hits the horizontal plane. It is given by,

$$t = \frac{2u \sin \theta}{g}$$

where u = initial velocity of the projectile

θ = angle which the projectile makes with the horizontal.

Horizontal range : It is the maximum horizontal distance between the point of projection and the point in the horizontal plane where the projectile hits. It is given by,

$$R = \frac{u^2 \sin 2\theta}{g}$$

Maximum range, $R_m = \frac{u^2}{g}$

(at $\sin 2\theta = 1$ or $\theta = 45°$)

Maximum height : It is the maximum vertical distance covered by the projectile above the horizontal plane. It is given by,

$$H = \frac{u^2 \sin^2 \theta}{2g}.$$